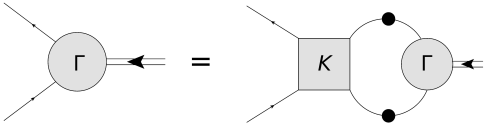
Solutions to the Bethe-Salpeter Equation via Integral Representations



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Introduction

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   - Nakanishi Forms
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Dyson-Schwinger Equations

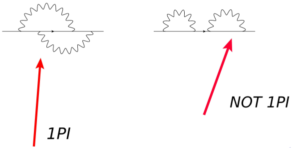
Dressed 2-point function for quarks

$$iS(x-y) = \frac{\langle 0_H | T\psi_H(x)\overline{\psi}_H(y)|0_H\rangle}{\langle 0_H|0_H\rangle} = \langle 0_I | Te^{i\int \mathcal{L}_I}\psi_I(x)\overline{\psi}_I(y)|0_I\rangle_c$$

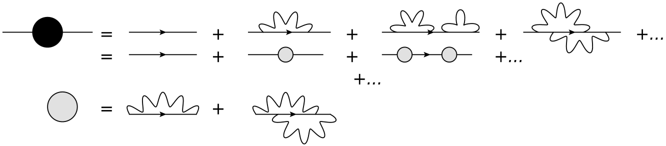
1. Wick's theorem and perturbation theory:
   $iS(x-y) = \sum_n^\infty \frac{i^n}{n!}\int d^4z ... \langle O| T\mathcal{L}_{int}(z_1)...\psi(x)\overline{\psi}(y)|0\rangle_c$
2. Non-perturbative approach

One-Particle Irreducible (1PI) Diagrams

Any diagram that cannot be split in two by removing a single line; let the sum of all 1PI diagrams be called self-energy

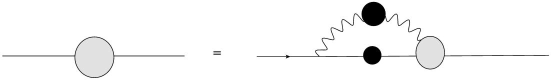


DSE Continued



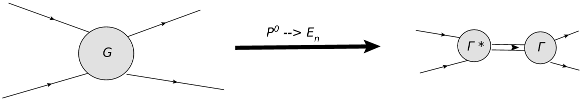
$$
\begin{aligned}
S &= S_0 + S_0 \Sigma S_0 + S_0 \Sigma S_0 \Sigma S_0 + \ldots \\
&= S_0 + S_0 \Sigma (S_0 + S_0 \Sigma S_0 + \ldots) \\
&= S_0 + S_0 \Sigma S
\end{aligned}
$$

$$
\rightarrow S^{-1} = S_0^{-1} + \Sigma
$$



Bound States and Bethe-Salpeter Equation

- Bound states appear as poles in scattering amplitudes (on real axis for stable states, below for unstable states)[1]
- By definition: $H|P,t\rangle = P^0|P,t\rangle \rightarrow |P,t\rangle = e^{-iP^0 t}|P,0\rangle$
- So the bound state contribution to a completeness sum in an amplitude is $\langle f, t_f|P,t\rangle\langle P,t|i,t_i\rangle = \langle f|P\rangle e^{-i(t_f-t_i)E}\langle P|i\rangle$
- The fourier transform $t_f - t_i \Rightarrow p^0$ generates a pole at $p^0 = E - i\epsilon$
- $\int dte^{-i(E-p_0)t}|\Gamma^*\rangle\langle\Gamma| = \frac{|\Gamma^*\rangle\langle\Gamma|}{p_0 - E + i\epsilon}$



[1] Paul Hoyer - Bound states - from QED to QCD arXiv: 1402.5005v1

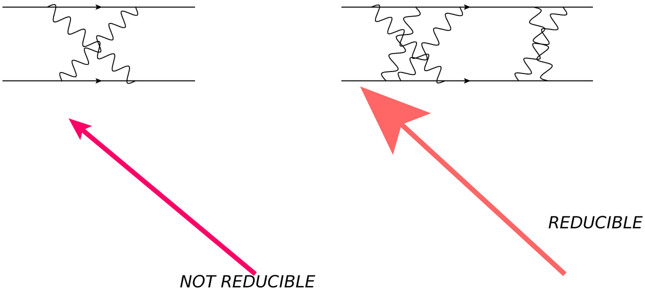
Derivation of Bethe-Salpeter Equation

In order to find an expression for Γ , consider the (above) 4-pt function:

Two-Particle Irreducible (2PI) Diagrams

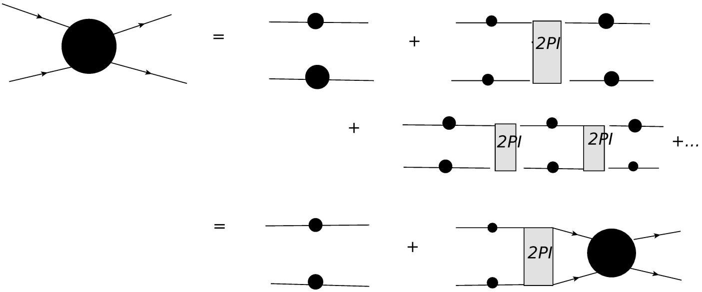
Any diagram that cannot be split in two by removing two (internal fermion) lines [a]

[a] Huang, Quantum Field Theory Ch10

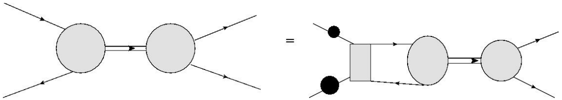


cont.

So the series for the four-point function is:

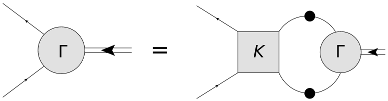


Becomes:



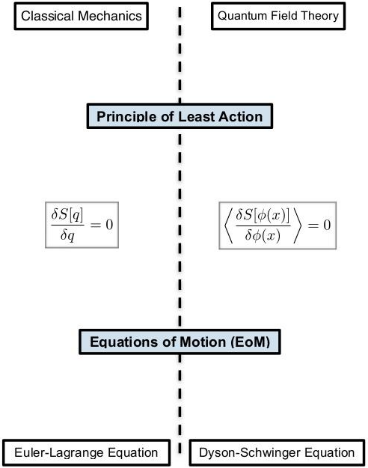
BSE

Assume the form $G \to \frac{\Gamma^*(P)\Gamma(P)}{P^0 - E_P}$



$$[\Gamma(p;P)]_{tu} = \lambda \int \frac{d^4k}{(2\pi)^4} K_{tu}^{rs}(p,k;P)\,[S(k_+)\Gamma(k;P)S(k_-)]_{sr}\ .$$

Functional Derivation - Quick Look



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[2] Taken from S.Qin's (ANL) APS talk Spring 2015

Functional Derivation

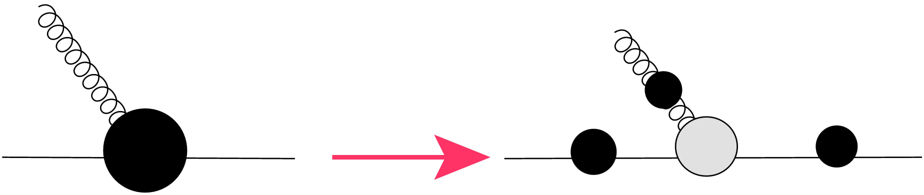
$$
\begin{aligned}
0 &= \int D\mu \frac{\delta}{\delta \overline{q}(x)} e^{-S_{QCD}+S_{sources}} \qquad S_{sources} \sim (\overline{\xi}, q) \\
&= \int \left(\frac{\delta S}{\delta \overline{q}(x)}[q, \overline{q}, w, \overline{w}, A] + \xi\right) e^{-S_{QCD}+S_{sources}} \\
&= \left(-\frac{\delta S}{\delta \overline{q}(x)}\left[\frac{\delta}{\delta \overline{\xi}}, \ldots\right] + \xi\right) Z[\overline{\xi}, \xi, \overline{\eta}, \eta, \lambda]
\end{aligned}
$$

Take another functional derivative and divide by Z:

$$
0 = -(\gamma_\mu \delta_\mu + m + igt^a \gamma_\mu \frac{\delta}{\delta \lambda^a_\mu}) \frac{\delta^2}{\delta \overline{\xi}(x) \delta \xi(y)} logZ + \delta(x-y)
$$

Legendre transformation:

$\Gamma[q, \overline{q}, w, \overline{w}, A] \doteq \int (\overline{\xi} q + \overline{q} \xi + \lambda A + \overline{\eta} w + \overline{w} \eta) - logZ$



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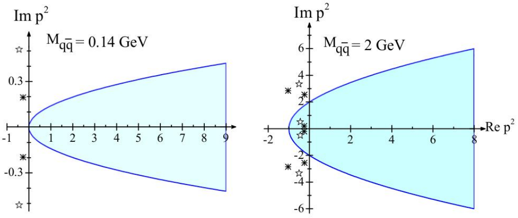
[3]Great discussion of effective action in Weinberg The Quantum Theory of Fields Vol II Ch 16

[4]full derivation in: Marco Viebach - Diplomarbeit - Dyson-Schwinger equation for the quark propagator at finite temperatures

[5]see also Ch10 of Itzykson+Zuber

Solving some things

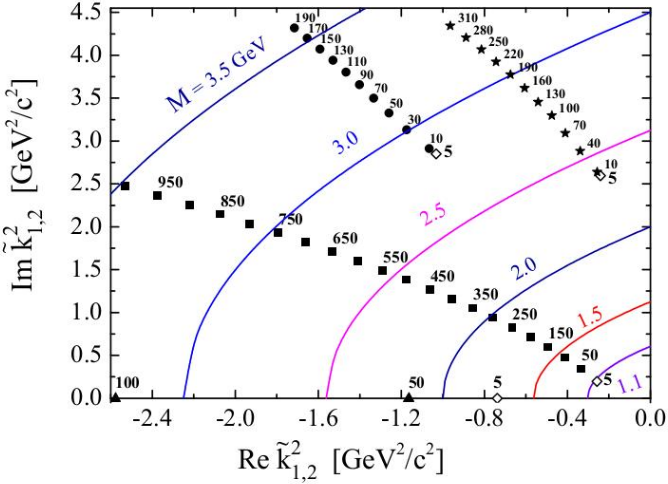
In the BSE, $(k^{\pm})^2 = k^2 + P^2/4 \pm 2k \cdot q = k^2 - M^2/4 \pm ikMx$, if $P = (iM, \vec{0})$ in Euclidean metric [6]



These poles suggest the propagators can be represented as:

$$S(p) = \sum_{n=1}^{N} \left\{ \frac{z_n}{i\not{p} + m_n} + \frac{z_n^*}{i\not{p} + m_n^*} \right\}$$

[6] Analytical properties of the quark propagator from a truncated Dyson-Schwinger equation in complex Euclidean space, Dorkin et al., PhysRevC.89.034005



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[7]Dorkin et al. arxiv:1412.3345v1

Fit to Numerical Solution

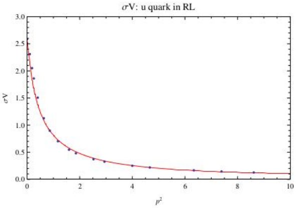


Figure 1: σ_V for u quark in RL

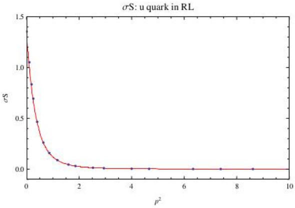
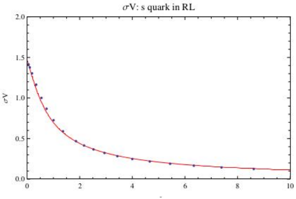
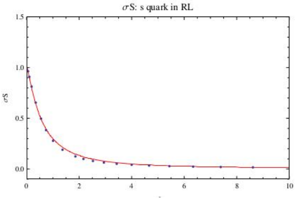


Figure 2: σ_S for u quark in RL





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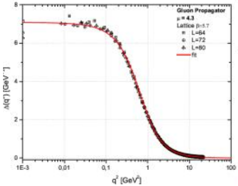
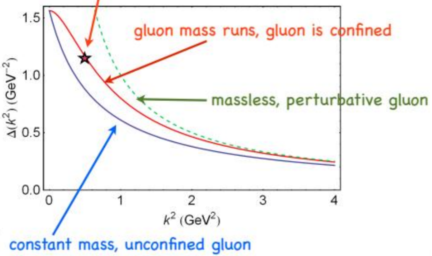
[8] Thanks to Shi Chao, Nanjing University

QCD: Quarks and Gluons are Confined (postulate)

- Only color singlet states in the spectrum of Hqcd
- Allowed states in Euclidean field theory satisfy certain criteria
- One of which is the norm or spectral density must be positive, definite
- Osterwalder-Schrader axium 3 [Comm. Math. Phys. v42, 281 (1975)+]
- Latter is violated if there is an inflection pt in 2-pt fn vs k^2





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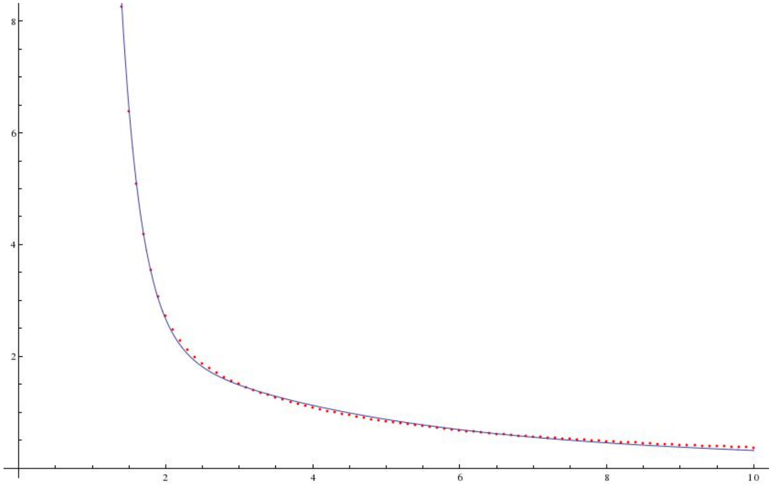
[9] slide from P.Tandy

Going Further - Kernel

Fit the LR-kernel to a similar form: $D(k^2) = \sum \frac{Z_i}{k^2+m_i^2} + \frac{Z_i^*}{k^2+(m_i^*)^2}$



Fit Existing BSE Ampls, DSE solns for S(k) for Feyn Integral Method

$$\Gamma_\pi(q^2, q\cdot P) = \gamma_5 \{E_\pi(q^2, q\cdot P) + \not{P}\, F_\pi(..) + \not{q}\, q\cdot P\, G_\pi(..) + \sigma : qP\, H_\pi(..)\}$$

Use Nakanishi Representation (1965) :- $\mathcal{F} = E,\ F,\ G,\ \text{or}\ H$

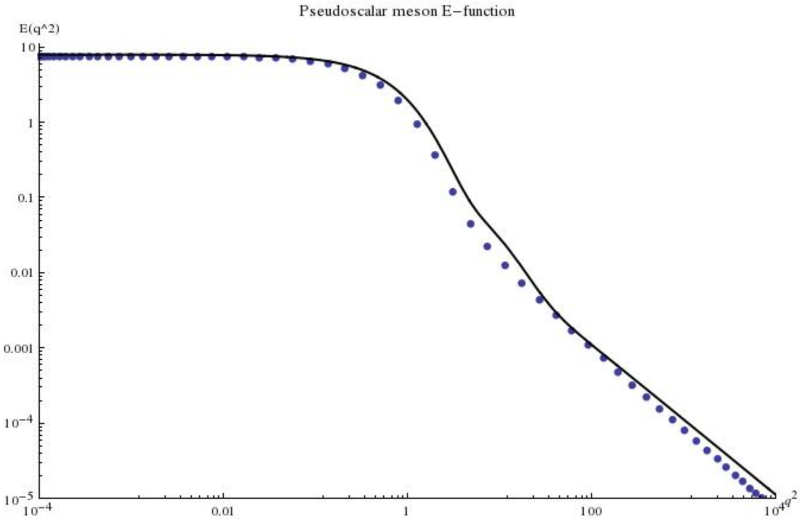
$$\mathcal{F}(q^2; q\cdot P) = \int_{-1}^{1} d\alpha \int_0^\infty d\Lambda \left\{ \frac{\rho_{IR}(\alpha;\Lambda)}{(q^2 + \alpha q\cdot P + \Lambda^2)^{m+n}} + \frac{\rho_{UV}(\alpha;\Lambda)}{(q^2 + \alpha q\cdot P + \Lambda^2)^{n}} \right\}$$

cf. 7.1 of Peskin+Schroeder, where $\langle\Omega| T\phi(x)\phi(y)|\Omega\rangle$ is written in integral (Kallen-Lehmann spectral rep) form; ex: do the same for $\langle\Omega| T\phi(z)\phi(x)\phi(y)|\Omega\rangle$

- Now all of the elements of the BSE have the form $\frac{Num}{p^2+M^2}$
- Use Feynman parameters:

$$\frac{1}{A_1^{m_1}A_2^{m_2}...A_n^{m_n}} = \int_0^1 dx_1...dx_n \delta(\sum x_i - 1) \frac{\prod x_i^{m_i-1}}{\left[\sum x_i A_i\right]^{\sum m_i}} \frac{\Gamma[m_1 + ...m_n]}{\Gamma(m_1)...\Gamma(m_n)} \quad (1)$$

- The result is a known Euclidean integral, and hence are left with a reduced-dimensional integral over Feynman parameters (poles are therefore not a problem)
- Eigenvalue equation $\lambda(P^2)E(k,P) = \int_q K(k-q)S(q_+)E(q,P)S(q_-)$ with solution $\lambda(P^2 = -M^2) = 1$


Pseudoscalar meson E−function
E(q^2)
10
1
0.1
0.01
0.001
10^{-4}
10^{-5}
10^{-4}
0.01
1
100
10^4
q^2

Looking ahead

charm + bottom physics, investigation of exited hadrons, em form factors in timelike region, medium- and large-Q^2 spacelike form factors, finite temp., etc.

- Also, thanks to HUGS organizers, presenters, JLab staff, etc.